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Decay of a Magnetohydrodynamic Shock Wave Produced by a Piston

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It is the purpose of this note to discuss the interaction of a uniform shock wave and a centered rarefaction wave in the one-dimensional, unsteady flow of an ideal, inviscid, perfectly conducting, monatomic, compressible fluid subjected to a transverse magnetic field. Let a uniform shock wave be generated by pushing a piston with constant velocity into a tube filled with gas at rest, and, at some later time, let the piston be stopped abruptly thus generating a centered rarefaction wave, which propagates with true-sonic speed and ultimately overtakes the shock and diminishes its strength. Under the assumption that the shock wave is weak or at most of moderate strength, the width of the interaction zone and the subsequent motion of the shock wave may be determined by the use of the author's extension[1] of the Friedrichs' theory[2, 3] for conventional gasdynamic flows. For a monatomic fluid, generalized Riemann invariants[4] and the flow parameters in a centered simple wave[5] may be determined explicitly. These facts form the basis for a simple extension of the Friedrichs' theory to the magnetic case, and, in the limit of vanishing magnetic field, the extended theory reduces exactly to that derived by Friedrichs.

Let u, c, B, μ, ρ, $b^2 = B^2/\mu\rho$, $\omega = (b^2 + c^2)^{1/2}$, $\Omega = u + \omega$, $m = b/c$, and U be, respectively, the particle velocity, local speed of sound, magnetic induction, permeability, density, square of the Alfvén speed, the true-sonic speed, slope of the forward-facing characteristics of the simple wave, a measure of the applied field, and shock velocity. Let the flow in front of the shock be denoted by the subscript zero, and let the subscript one denote the flow behind the shock or in front of the rarefaction wave. Consequently, the velocity of the piston is u_1, and the shock velocity is U_1. Further, it is assumed that the piston is stopped at time t_R at the position $x_R = u_1 t_R$.

For the magnetic case, the generalized Riemann invariants are[4]

$$u/2 + (\omega/c)^3/k = \alpha$$

$$-u/2 + (\omega/c)^3/k = \beta$$

where k is constant, and the Friedrichs' theory assumes $\beta_0 = (\omega_0/c_0)^3/k$ remains constant across the shock, i.e., the shock transition is replaced by the transition through the corresponding compression simple wave.

The strength of the shock may be characterized by the function $\sigma(\xi)$, defined by $\Omega - \omega_0 = \omega_0\sigma(\xi)$, where ξ is a parameter constant along a rectilinear characteristic of the simple wave. For a monatomic fluid, u and ω may be determined explicitly in terms of Ω, and, in particular, since ω_1 may be expressed in terms of u_1, the piston velocity, it suffices to state that

$$\sigma_1 = [u_1 + \omega_1(u_1) - \omega_0]/\omega_0 \tag{1}$$

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which is known. Then the shock velocity may be determined from the formula[1]

$$U_1 = \omega_0[1 + \sigma_1/2 + D\sigma_1^2/8] \tag{2}$$

where

$$D = [69m_1^4 + 136m_1^2 + 64 - 4m_1^2(1 + m_1^2)^{3/2}]/(8 + 9m_1^2)^2$$

and

$$\lim_{m_1 \to 0} D = 1$$

The centered rarefaction wave may be described by $x = x_R + \Omega(t - t_R)$, and its strength is determined by the condition that the flow behind it, characterized by the subscript two, is at rest. Consequently, $u_2 = 0$, and it follows that $\sigma_2 = 0$ and $\omega_2 = \omega_0$, so that the motion of the tail of the rarefaction wave is given by

$$x = x_T = x_R + \omega_0(t - t_R)$$

The head of the rarefaction wave catches up with the shock at a time t_1 at a position

$$x_1 = U_1 t_1 = x_R + (u_1 + \omega_1)(t_1 - t_R) \tag{3}$$

From Eqs. (1–3), it follows that

$$t_1 = 8\omega_1 t_R/\omega_0\sigma_1(4 - D\sigma_1)$$

The solution for the shock path obtained when a weak shock interacts with a rarefaction wave described by

$$x = \xi + \Omega(\xi)t$$
$$\Omega(\xi) = u(\xi) + \omega(\xi)$$
$$-u(\xi)/2 + [\omega(\xi)/c(\xi)]^3/k = \beta_0$$

is given by

$$\omega_0 t(\xi) = 8\left[\frac{4 - D\sigma_1(\xi)}{\sigma_1(\xi)}\right]^2 \int_\xi^0 \frac{\sigma_1(y)dy}{[4 - D\sigma_1(y)]^3} \tag{4}$$

In order to use Eq. (4) for the present problem in which the interaction begins at $t = t_1$, it is convenient to describe the simple wave with respect to a new origin (x_1, t_1). This gives

$$x - x_1 = \omega_0(1 + \sigma)(t - t_1) + \omega_0(t_1 - t_R)(\sigma - \sigma_1) \tag{5}$$

From Eq. (5), it follows that $\xi = -\omega_0(t_1 - t_R)(\sigma_1 - \sigma)$, and substituting into Eq. (4), which is the magnetohydrodynamic generalization of Eq. (8.6) of Ref. 2, the following result is obtained:

$$t \equiv t_s = t_1 + \frac{8(t_1 - t_R)}{\sigma^2(4 - D\sigma_1)^2}[\sigma^2(D\sigma_1 - 2) - D\sigma_1^2\sigma + 2\sigma_1^2] \tag{6}$$

Therefore,

$$x \equiv x_s = x_R + \omega_0(1 + \sigma)(t - t_R) \tag{7}$$

Equations (6–7) give the parametric representation of the motion of the shock. In order to describe this motion by giving x as a function of t, Eq. (6) may be inverted to give $\sigma = \sigma(t)$ which, upon substitution into Eq. (7), gives $x = x(t)$. From the asymptotic form of this result, valid for large t and consequently for small σ, it may be seen, exactly as in the conventional gasdynamic case, that the width of the decaying shock wave, i.e., the distance between the shock front and the tail of the rarefaction wave, increases in direct proportion to the square root of the time elapsed.

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Optimum Translation and the Brachistochrone

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1. Introduction

Requirements for a manned lunar landing include the ability to descend to a low altitude, hover for a time sufficient to examine the potential landing site, and translate if necessary to a new landing site. In Ref. 1, several possible translation maneuvers are analyzed from the point of view of fuel consumption. In this paper we determine the maneuver which results in the least possible fuel consumption. The solution is effected by transforming the problem into a form equivalent to the brachistochrone with limited slope and applying standard techniques of the calculus of variations.

It will be assumed that the maneuver is horizontal and perpendicular to the gravitational acceleration g, which is constant in magnitude and direction. It will further be assumed that the thrust acceleration a_T is bounded in magnitude. In particular, let μ denote the horizontal component of the thrust to weight ratio a_T/g and M the maximum value of μ. Since for horizontal motion the vertical component of the thrust acceleration is g, the total acceleration due to thrust is limited as follows:

$$0 \leqslant a_T \leqslant g(M^2 + 1)^{1/2} \tag{1}$$

The assumption of the limit on thrust acceleration, rather than thrust, results in analytical simplifications which enable closed form solutions to be obtained easily.

2. The Variational Problem

The equations of motion are

$$\ddot{x} = \mu g \qquad \ddot{y} = 0 \tag{2}$$

where g is the vertical component of the thrust acceleration and μg is the horizontal component. The thrust acceleration magnitude is then

$$a_T = g(1 + \mu^2)^{1/2} \tag{3}$$

and the characteristic velocity is

$$\Delta V = \int_0^{t_f} a_T\, dt = g \int_0^{t_f} (1 + \mu^2)^{1/2}\, dt \tag{4}$$

where t_f is the time of flight.

The problem is to determine that function $\mu(t)$ which minimizes (4) subject to (1), (2), and the end conditions:

$$\begin{aligned} t = 0: &\qquad x = 0 \qquad \dot{x} = 0 \\ t = t_f: &\qquad x = D \qquad \dot{x} = 0 \end{aligned} \tag{5}$$

That is, the vehicle translates through the distance D, beginning and ending in a stationary hovering position.

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The problem can be transformed into the classical first problem of the calculus of variations by treating distance as the independent and velocity as the dependent variable. We define $\xi = 2x/D$, $u = v^2/gD$, $v = \dot{x}$ and Eq. (4) becomes

$$\frac{\Delta V}{(gD)^{1/2}} = \frac{1}{2}\int_0^2 \left[\frac{1 + u'^2}{u}\right]^{1/2} d\xi \tag{6}$$

where $u' = du/d\xi = \mu$. The inequality constraint (1) can be expressed by the equality,[2]

$$\epsilon^2 = M^2 - u'^2 \tag{7}$$

where ϵ is a new state variable. Incorporating the constraint (7) by the method of Lagrange multipliers leads to the following integral to be minimized:

$$\frac{\Delta V}{(gD)^{1/2}} = \int_0^1 \left[\left(\frac{1 + u'^2}{u}\right)^{1/2} + \lambda(\epsilon^2 - M^2 + u'^2)\right] d\xi = \int_0^1 F(x,u,u')d\xi \tag{8}$$

where the integration interval has been reduced to half the distance to take advantage of symmetry. The boundary conditions are

$$\begin{aligned} \xi = 0: &\qquad u = 0 \\ \xi = 1: &\qquad u \text{ not prescribed} \end{aligned} \tag{9}$$

That is, the problem of translating a distance D, beginning and ending at zero velocity, is equivalent to that of translating a distance $D/2$ beginning at rest and with arbitrary final velocity.

Equations (8) and (9) are identical to those obtained from the brachistochrone problem, where it is desired to find the shape $y(x)$ of a frictionless wire joining two points, such that a particle sliding along the wire under the influence of gravity alone will travel between the points in minimum time. To complete the analogy $\xi \equiv 2x/D$, $u \equiv 2y/D$ and $\Delta V \equiv gT$, where T is the duration of the motion. The constraint corresponds to a limit on the maximum slope the wire is allowed to assume.

3. Solution of the Variational Problem

The solution is obtained by direct application of the methods of the calculus of variations described in Ref. 2. In particular it may be shown that either $\epsilon = 0$ and the acceleration is maximum or minimum, or $\lambda = 0$ and an intermediate level prevails. The corner conditions show that λ and u' are continuous at the transition and the free boundary conditions yield $u' = 0$ at $\xi = 1$, where the velocity is maximum. The transition from maximum to intermediate thrust acceleration is determined from the Weierstrass condition.

The resulting optimum thrust acceleration is given as a function of the instantaneous velocity by

$$\begin{aligned} \mu = u' &= M &&(u < u_s) \\ &= \left(\frac{u_m}{u} - 1\right)^{1/2} &&(u > u_s) \end{aligned} \tag{10}$$

where the maximum value of the velocity variable u is

$$u_m = \left[\frac{1}{M} + \cot^{-1}\frac{1}{M}\right]^{-1} \tag{11}$$

and the switching condition is

$$u_s = u_m/(1 + M^2) \tag{12}$$

The trajectory is

$$\begin{aligned} \xi &= u/M \qquad (u < u_s) \\ &= 1 - u_m \left\{\left[\frac{u}{u_m}\left(1 - \frac{u}{u_m}\right)\right]^{1/2} + \cos^{-1}\left[\frac{u}{u_m}\right]^{1/2}\right\} \qquad (u > u_s) \end{aligned} \tag{13}$$